

Selected Aspects of Soliton Theory Constant boundary conditions

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Abstract

Brief review of the methods for solving the multicomponent nonlinear Schrödinger (MNLS) equations and analysis of their Hamiltonian structures is given. Main attention is paid to the MNLS related to the *C.II*- and *D.III*-types symmetric spaces with nonvanishing (constant) boundary conditions. The spectral properties of their Lax operators are described. The derivation of the trace identities is outlined. The involutivity of their integrals of motion is proved using the method of the classical *R*-matrix.

1 Introduction

The integrability of the well known (scalar) NLS eq.:

$$iq_t + q_{xx} + 2|q(x, t)|^2 q(x, t) = 0, \quad (1)$$

was discovered by Zakharov and Shabat in their pioneer paper [1] which strongly stimulated the search of other important integrable nonlinear evolution equations (NLEE).

Soon after [1] Zakharov and Shabat proved the integrability and physical importance of the NLS with constant boundary conditions [2] (CBC):

$$iq_t + q_{xx} - 2(|q(x, t)|^2 - \rho^2)q(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} q(x, t) = q_{\pm}, \quad (2)$$

where the asymptotic values q_{\pm} satisfy $|q_{\pm}|^2 = \rho^2$. Note the sign difference in the cubic nonlinearity as well as the additional term with the chemical potential. These changes are important from both physical and mathematical point of view.

Both versions of the NLS eq. served as paradigms on which the ISM was developed [1, 2, 3]. They served also as a tool for the development of the quantum ISM, see the review paper [4].

The simplest nontrivial multicomponent generalizations of NLS is the vector NLS eq. known also as the Manakov model [5]:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger \vec{q})\vec{q}(x, t) = 0, \quad (3)$$

where $\vec{q}(x, t)$ is an n -component complex-valued vector tending to zero fast enough for $x \rightarrow \pm\infty$. The CBC version of the vector NLS equations (especially the one with $n = 2$)

$$i\vec{q}_t + \vec{q}_{xx} - 2((q^\dagger \vec{q}) - \rho^2) \vec{q}(x, t) = 0, \quad (4)$$

also finds applications in nonlinear optics, plasma physics etc. Here $\lim_{x \rightarrow \pm\infty} \vec{q} = \vec{q}_\pm$, $\vec{q}_- = U_0 \vec{q}_+$ where U_0 is constant unitary matrix.

The close relations between the sectional curvatures of the symmetric spaces and the interaction constants of the integrable multicomponent NLS (MNLS) was discovered in [6] for vanishing boundary conditions (VBC). The basic ideas for constructing their soliton solutions via the dressing Zakharov-Shabat method were formulated in [7]. These ideas and results were developed in a number of more recent monographs [8] and papers, see [9, 10, 11, 12, 13, 14, 15, 16] and the numerous references therein.

However the properties of the MNLS with CBC have substantial differences as compare to the VBC case. This is due to the facts that: i) the corresponding phase space \mathcal{M} spanned by the allowed potentials is nonlinear; ii) the relevant Lax operators may have rather involved spectral properties. Both these facts, noticed long ago [17, 18, 19] (see also [20, 21]) explain why MNLS with CBC have not been so well studied.

In the next Section 2 we provide the necessary facts from the theory of MNLS and the relevant symmetric spaces [6] for vanishing boundary conditions. In Section 3 we reformulate these results for CBC. In particular we describe the spectral properties of MNLS with CBC for several types of symmetric spaces. In Section 4 we detail two examples of MNLS related to the algebras $sp(4)$ and $so(8)$. In the last Section 5 we concentrate on the Hamiltonian properties and their classical R -matrix formulation. In particular we outline how the integrals of motion from the fundamental series can be regularized so that they become regular functionals on the non-linear phase space \mathcal{M} .

2 MNLS with vanishing BC

Equations (1) and (3) are particular cases of the matrix NLS eq. which is obtained from the system:

$$\begin{aligned} i\mathbf{q}_t + \mathbf{q}_{xx} + 2\mathbf{q}\mathbf{r}\mathbf{q}(x, t) &= 0, \\ -i\mathbf{r}_t + \mathbf{r}_{xx} + 2\mathbf{r}\mathbf{q}\mathbf{r}(x, t) &= 0. \end{aligned} \quad (5)$$

with appropriate reductions (involution). Here $\mathbf{q}(x, t)$ and $\mathbf{r}^T(x, t)$ are $n \times m$ matrix-valued functions of x and t with $n > 1$, $m > 1$ which are smooth enough and tend to zero fast enough for $x \rightarrow \pm\infty$. The best known involution compatible with the evolution of (5) is

$$\mathbf{r} = B_- \mathbf{q}^\dagger B_+^{-1}, \quad B_\pm = \text{diag}(\epsilon_1^\pm, \dots, \epsilon_m^\pm), \quad (\epsilon_s^\pm)^2 = 1. \quad (6)$$

and the corresponding MNLS equation is of the form:

$$i\mathbf{q}_t + \mathbf{q}_{xx} + 2\mathbf{q}B_-\mathbf{q}^\dagger B_+^{-1}\mathbf{q}(x, t) = 0 \quad (7)$$

For $n = m = 1$ and $r = \epsilon q^*$ the system (5) goes into the scalar NLS equation; for $m = 1$ and $n > 1$ and with appropriate choice of the involution (6) eq. (5) can be transferred into the Manakov model or into eq. (7).

The MNLS (7) is known to be closely related to the symmetric spaces [6]. All these versions of NLS are solvable by applying the ISM to a generalization of the Zakharov-Shabat system of the form:

$$L\psi \equiv \left(i \frac{d}{dx} + Q(x, t) - \lambda J \right) \psi(x, \lambda) = 0, \quad (8)$$

$$M\psi \equiv \left(i \frac{d}{dt} + V_0(x, t) + \lambda V_1(x, t) - 2\lambda^2 J \right) \psi(x, \lambda) = 0, \quad (9)$$

$$Q(x, t) = \begin{pmatrix} 0 & \mathbf{q}(x) \\ \mathbf{r}(x) & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (10)$$

where $Q(x, t)$ and J are $(n + m) \times (n + m)$ matrices with compatible block structure and $V_0(x, t)$, $V_1(x, t)$ are expressed in terms of Q and its x -derivative:

$$V_1(x, t) = 2Q(x, t), \quad V_0(x, t) = -[Q, \text{ad}_J^{-1}Q] + 2i \text{ad}_J^{-1}Q_x. \quad (11)$$

An effective tool to obtain new versions of MNLS type equations is the reduction group introduced by Mikhailov [22]. It allows one to impose algebraic constraints on $Q(x, t)$ which are automatically compatible with the evolution. For example, the involution (6) (or \mathbb{Z}_2 -reduction) can be written as:

$$BU^\dagger(x, t, \lambda^*)B^{-1} = U(x, t, \lambda), \quad B = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}, \quad (12)$$

where B is an automorphism of \mathfrak{g} , i.e. matrix such that $B^2 = \mathbb{1}$, $[J, B] = 0$ and:

$$U(x, t, \lambda) = Q(x, t) - \lambda J. \quad (13)$$

Reductions leading to new types of MNLS systems are demonstrated in [14, 23].

The spectral properties of the Lax operators (8) as well as the basic properties of the MNLS equations for the A -type symmetric spaces are analyzed in [8, 14, 10]. Our aim below is to extend these results to the MNLS equations with CBC for the C - and D -type symmetric spaces.

3 MNLS with CBC

Though similar in form, the MNLS with CBC require a number of important changes. These changes are based on two constraints which ensure: i) regular behaviour of the solutions for $t \rightarrow \pm\infty$; in other words we want to avoid strong

oscillations for large times; ii) require that the spectrum of the two asymptotic operators $L_{\pm} = id/dx + U_{\pm}(\lambda)$ have the same spectrum. Here

$$U(x, t, \lambda) = Q(x, t) - \lambda J, \quad U_{\pm}(\lambda) \equiv \lim_{x \rightarrow \pm\infty} U(x, t, \lambda) = Q_{\pm} - \lambda J. \quad (14)$$

The first requirement can be satisfied by regularizing the MNLS, i.e. by conveniently adding linear in \mathbf{q} terms, see eq. (15). The corresponding regularized MNLS have the form:

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2\mathbf{q}\mathbf{q}^{\dagger}\mathbf{q} + \mathbf{q}\mu + \bar{\mu}\mathbf{q} = 0, \quad (15)$$

with the boundary conditions

$$\lim_{x \rightarrow \pm\infty} \mathbf{q}(x, t) = \mathbf{q}_{\pm}, \quad \mu = \mathbf{q}_+^{\dagger}\mathbf{q}_+ = \mathbf{q}_-^{\dagger}\mathbf{q}_-, \quad \bar{\mu} = \mathbf{q}_+\mathbf{q}_+^{\dagger} = \mathbf{q}_-\mathbf{q}_-^{\dagger}, \quad (16)$$

The second requirement will be satisfied limiting values \mathbf{q}_{\pm} must satisfy also

$$Q_+^2 = Q_-^2. \quad (17)$$

which ensures that $U_+(\lambda)$ and $U_-(\lambda)$ have the same sets of eigenvalues.

The M -operators of the MNLS with CBC (15) are given by (9) with

$$V_0(x, t) = -[Q, \text{ad } \bar{J}^{-1}Q] + 2i\text{ad } \bar{J}^{-1}Q_x(x, t) + [Q_{\pm}, \text{ad } \bar{J}^{-1}Q_{\pm}]. \quad (18)$$

The additional terms, as compare to (11) ensure the regular behavior of the solutions for large t .

Lax operators of the form (8) can be associated with each of the symmetric spaces described below. They are defined by specifying the simple Lie algebra \mathfrak{g} (having typical representation in matrices $N \times N$, $N = s + s'$) and J :

A.II $\mathfrak{g} \simeq A_{N-1} \equiv sl(N)$, $J = H_{\vec{a}}$, where the vector \vec{a} in the root space \mathbb{E}^r dual to J is given by $\vec{a} = \sum_{k=1}^s e_k - \sum_{k=s+1}^N e_k$;

The next two cases require that $s = s' = r$ and that $N = 2r$ is even.

C.II $\mathfrak{g} \simeq C_r \equiv sp(2r)$, $J = H_{\vec{a}}$, where the vector \vec{a} in the root space \mathbb{E}^r dual to J is given by $\vec{a} = \sum_{k=1}^r e_k$;

D.III $\mathfrak{g} \simeq D_r \equiv so(2r)$, $J = H_{\vec{a}}$, where the vector \vec{a} in the root space \mathbb{E}^r dual to J is given by $\vec{a} = \sum_{k=1}^r e_k$.

The spectrum of the asymptotic operators L_{\pm} is purely continuous and is determined by the the eigenvalues of Q_{\pm} which generically may be arbitrary complex numbers. The spectra of A -type symmetric spaces were described in [18]. For $C.II$ - and $D.III$ -type symmetric spaces the spectra may consist of the following types of branches, see the left panel of fig. 1.

a) $\nu_k \neq \pm\nu_k^*$, $k = 1, \dots, l_1$ - two branches of two-fold spectrum filling up the hyperbola's arcs $\text{Re } \lambda \text{Im } \lambda = \text{Re } \nu_k \text{Im } \nu_k$ on which $|\text{Re } \lambda| \geq |\text{Re } \nu_k|$;

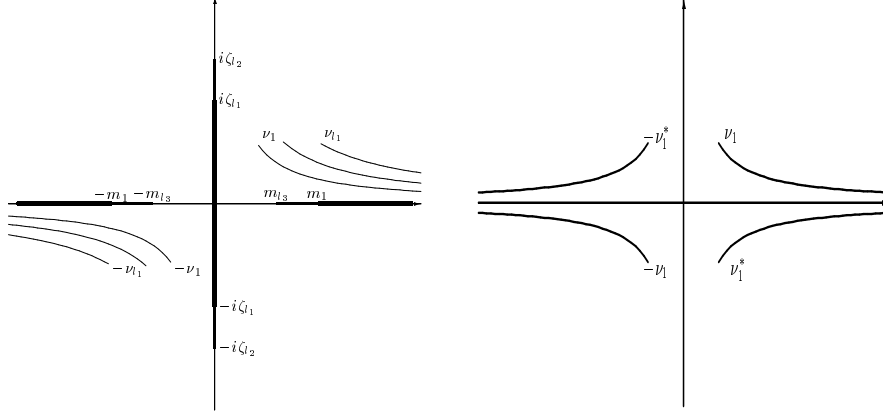


Figure 1: Left panel: the continuous spectrum of L , generic case; Right panel: the continuous spectrum of the $sp(4)$ and $so(8)$ MNLS with CBC for $D < 0$; the only difference is that while the multiplicity of the spectra of $sp(4)$ is 2 the one for $so(8)$ is 4.

- b) $\nu_{l_1+k} = -\nu_{l_1+k}^* = i\zeta_k$, $k = 1, \dots, l_2$ – two branches of two-fold spectrum filling up the real axis and the segment $|\text{Im } \lambda| \leq |\zeta_k|$ of the imaginary axis;
- c) $\nu_{l_1+l_2+k} = \nu_{l_1+l_2+k}^* = m_k$, $k = 1, \dots, l_3 = r - l_1 - l_2 + 1$ – two branches of two-fold spectrum filling up the segments $|\text{Re } \lambda| \geq |m_k|$ of the real axis;

In order to proceed with the C_r - and D_r -types symmetric spaces we will need to introduce their definitions and Cartan-Weyl basis in the typical representations. In what follows we will define the Lie algebra \mathfrak{g} by:

$$\mathfrak{g} \equiv \{X : X + S_0 X^T S_0^{-1} = 0\}, \quad (19)$$

where

$$S_0 = \sum_{s=1}^r (-1)^{s+1} (E_{s\bar{s}} - E_{\bar{s}s}), \quad \text{for } \mathfrak{g} \simeq sp(2r), \quad (20)$$

$$S_0 = \sum_{s=1}^r (-1)^{s+1} (E_{s\bar{s}} + E_{\bar{s}s}), \quad \text{for } \mathfrak{g} \simeq so(2r). \quad (21)$$

Here $\bar{s} = 2r + 1 - s$ and E_{ks} are $2r \times 2r$ matrices defined by $(E_{ks})_{jl} = \delta_{kj} \delta_{sl}$. Note that $S_0^2 = \epsilon_0 \mathbb{1}$, where $\epsilon_0 = -1$ for $sp(2r)$ and $\epsilon_0 = 1$ for $so(2r)$.

By e_k , $k = 1, \dots, r$ we denote the vectors forming an orthonormal basis in the root spaces \mathbb{E}^r for both types of algebras. With the above definitions of \mathfrak{g} their Cartan generators H_k dual to e_k are diagonal and given by:

$$H_k = E_{kk} - E_{\bar{k}\bar{k}}, \quad (22)$$

As we already mentioned, the element J dual to \vec{a} is $J = \sum_{k=1}^r H_k$. Using J we can split the set of positive roots into two subsets $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ which for $\mathfrak{g} \simeq sp(2r)$ are:

$$\Delta_0^+ = \{e_i - e_j\}, \quad \Delta_1^+ = \{2e_i, \quad e_i + e_j\}, \quad 1 \leq i < j \leq r, \quad (23)$$

while for $\mathfrak{g} \simeq so(2r)$ have the form:

$$\Delta_0^+ = \{e_i - e_j\}, \quad \Delta_1^+ = \{e_i + e_j\}, \quad 1 \leq i < j \leq r. \quad (24)$$

The root vectors in the typical representation are given by:

$$E_{e_i - e_j} = E_{ij} - (-1)^{i+j} E_{\bar{j}\bar{i}}, \quad E_{e_i + e_j} = E_{i\bar{j}} - \epsilon_0 (-1)^{i+j} E_{j\bar{i}}, \quad (25)$$

where $1 \leq i \leq j \leq r$ and $\epsilon_0 = \pm 1$ as defined above. Note that for $sp(2r)$ $\epsilon_0 = -1$ and eq. (25) provides also the expressions for E_{2e_j} by putting $i = j$; for $so(2r)$ $\epsilon_0 = 1$ and putting $i = j$ in eq. (25) gives vanishing result which is compatible with the fact that $2e_j$ are not roots of $so(2r)$.

The continuous spectrum of L coincides with the spectra of L_{\pm} . There are no apriory restrictions on the locations of the discrete eigenvalues of L (8).

The construction of the fundamental analytic solution for the problem (8) is rather tedious. Of all configurations for the set of eigenvalues $\{m_1, \dots, m_s\}$ we consider only the generic one:

$$m_1 > m_2 > \dots > m_s > 0. \quad (26)$$

Cases when subsets of $\{m_k\}$ are equal can be considered analogously.

Let us outline the construction of the FAS. The first peculiarity is related to the fact that the spectrum multiplicity may vary with λ , see the left panel of fig. 1. This reflects on the definition of the Jost solutions. For the case when all eigenvalues of Q_{\pm} are real, i.e. of the type c) above we have:

$$\psi(x, \lambda) \xrightarrow{x \rightarrow \infty} \psi_0(\lambda) e^{-iJ(\lambda)x} P(\lambda), \quad \phi(x, \lambda) \xrightarrow{x \rightarrow -\infty} \phi_0(\lambda) e^{-iJ(\lambda)x} P(\lambda), \quad (27)$$

$$J(\lambda) = \sum_{k=1}^r j_k(\lambda) H_k, \quad P(\lambda) = \exp \left(2\pi i \sum_{k=1}^r P_k(\lambda) H_k \right), \quad (28)$$

$$j_k(\lambda) = \sqrt{\lambda^2 - m_k^2}, \quad P_k(\lambda) = \theta(|\operatorname{Re} \lambda| - m_k).$$

The x -independent matrices $\psi_0(\lambda)$ and $\phi_0(\lambda)$ take values in the corresponding group \mathfrak{G} and satisfy

$$U_+(\lambda) \psi_0(\lambda) = -\psi_0(\lambda) J(\lambda), \quad U_-(\lambda) \phi_0(\lambda) = -\phi_0(\lambda) J(\lambda), \quad (29)$$

and are of the form:

$$\begin{aligned} \psi_0(\lambda) &= \varphi_0^+ U_0(\lambda), & \phi_0(\lambda) &= \varphi_0^- U_0(\lambda), \\ \varphi_0^{\pm} &= \begin{pmatrix} \varphi_1^{\pm} & 0 \\ 0 & \varphi_2^{\pm} \end{pmatrix}, & U_0(\lambda) &= \begin{pmatrix} \hat{S}_1 \underline{A} S_1 & S_1 \underline{B} \\ \underline{B} S_1 & \underline{A} \end{pmatrix}, \end{aligned} \quad (30)$$

Here the $r \times r$ matrices $\underline{\phi}_i^\pm$, \underline{A} , \underline{B} , S_1 are given by:

$$\begin{aligned} \underline{A}_{kl} &= \sqrt{\frac{\lambda + j_k}{2j_k}} \delta_{kl}, & \underline{B}_{kl} &= \sqrt{\frac{\lambda - j_k}{2j_k}} \delta_{kl}, & S_1 &= \sum_{s=1}^r (-1)^{s+1} E_{s, r+1-s}, \\ q_\pm q_\pm^\dagger \underline{\varphi}_1^\pm &= -\underline{\varphi}_1^\pm \underline{m}^2, & q_\pm^\dagger q_\pm \underline{\varphi}_2^\pm &= -\underline{\varphi}_2^\pm \underline{m}^2, & \underline{m}_{kl} &= m_k \delta_{kl}. \end{aligned} \quad (31)$$

Usually only the first and last columns of the Jost solutions $\psi(x, \lambda)$ and $\phi(x, \lambda)$ are analytic in λ . Nevertheless, applying the method in [18] we can construct fundamental solutions on each of the sheets of the 2^s -sheeted Riemannian surface, related to the set of roots $j_k(\lambda)$; each sheet of this surface is determined by the set of signs of $\{\epsilon_k(\lambda) : \epsilon_k = \text{sign Im } j_k(\lambda)\}$. Let us outline the construction of $\chi^+(x, \lambda)$ analytic in the region of the sheet \mathcal{S}_1 where:

$$\mathcal{S}_1 : \quad \text{Im } j_1(\lambda) > \text{Im } j_2(\lambda) > \cdots > \text{Im } j_r(\lambda) > 0. \quad (32)$$

First we construct the Gauss decomposition of the scattering matrix:

$$T(\lambda) = T^-(\lambda) D^+(\lambda) \hat{S}^+(\lambda), \quad (33)$$

where

$$\begin{aligned} T^-(\lambda) &= \exp \left(\sum_{\alpha > 0} \tau_\alpha^-(\lambda) E_{-\alpha} \right), & S^+(\lambda) &= \exp \left(\sum_{\alpha > 0} \sigma_\alpha^+(\lambda) E_\alpha \right), \\ D^+(\lambda) &= \exp \left(\sum_{k=1}^r \frac{2\delta_k^+(\lambda)}{(\alpha_k, \alpha_k)} H_{\alpha_k} \right), \end{aligned} \quad (34)$$

where H_{α_k} are Cartan elements dual to the simple roots α_k of \mathfrak{g} . Then the solution $\chi^+(x, \lambda)$:

$$\chi^+(x, \lambda) = \psi(x, \lambda) T^- D^+(\lambda) = \phi(x, \lambda) S^+(\lambda) \quad (35)$$

is analytic¹ in λ on the sheet \mathcal{S}_1 . Skipping the details of the proof which will be published elsewhere we only give some additional usefull facts about $\chi^+(x, \lambda)$.

First, the function $D^+(\lambda)$ is also analytic function of λ in \mathcal{S}_1 which generates the integrals of motion for the MNLS. Using the properties of the fundamental representations of the C_r and D_r series we have:

$$\langle \omega_j | T(\lambda) | \omega_j \rangle = \langle \omega_j | D^+(\lambda) | \omega_j \rangle = \exp((\omega_j, \vec{\delta}^+(\lambda))), \quad (36)$$

where ω_j is the j -th fundamental weight of \mathfrak{g} and $\vec{\delta}^+(\lambda) = \sum_{k=1}^r \delta_k^+(\lambda) e_k$. Note that the simple roots α_k and the fundamental weights ω_j satisfy the relation $2(\omega_j, \alpha_k)/(\alpha_k, \alpha_k) = \delta_{jk}$. More specifically for our examples we have:

$$\delta_1^+(\lambda) = \ln T_{11}(\lambda), \quad \delta_2^+(\lambda) = \ln \left\{ \begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix} \right\}_{T(\lambda)}, \quad (37)$$

¹More precisely it is not the functions $\chi^+(x, \lambda)$ that is analytic in λ but $\chi^+(x, \lambda) e^{iJ(\lambda)x}$.

for $sp(4)$ and

$$\begin{aligned}\delta_1^+(\lambda) &= \ln T_{11}(\lambda), & \delta_2^+(\lambda) &= \ln \left\{ \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right\}_{T(\lambda)}, \\ \delta_3^+(\lambda) &= \ln \left\{ \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right\}_{T(\lambda)} - \delta_4^+(\lambda), & \delta_4^+(\lambda) &= \frac{1}{2} \ln \left\{ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array} \right\}_{T(\lambda)},\end{aligned}\quad (38)$$

for $so(8)$. Here by $\left\{ \begin{array}{ccc} 1 & \dots & k \\ 1 & \dots & k \end{array} \right\}_{T(\lambda)}$ we denote the upper principal minor of order k of the scattering matrix $T(\lambda)$.

Note that due to the orthogonal symmetry inherent in D_4 all functions $\exp(\delta_k^+(\lambda))$ are polynomial expressions in terms of the matrix elements of $T(\lambda)$.

Fundamental analytic solutions can be constructed in analogous way on each of the sheets of the Riemannian surface.

4 Examples of MNLS with CBC on symmetric spaces

The requirement that $U(x, t, \lambda)$ and $V(x, t, \lambda)$ belong to \mathfrak{g} (see eq. (19)) can be formulated as the reduction condition:

$$\begin{aligned}S_0^{-1}U^T(x, t, \lambda)S_0 &= -U(x, t, \lambda), & S_0^{-1}JS_0 &= -J, \\ S_0^{-1}V^T(x, t, \lambda)S_0 &= -V(x, t, \lambda),\end{aligned}\quad (39)$$

which has trivial action on λ . Such reduction imposes restrictions only on the coefficients of $Q(x, t)$ so that for C_r we can put:

$$Q(x, t) = \sum_{i < j} (q_{ij}E_{e_i+e_j} + r_{ij}E_{-e_i-e_j}) + \frac{1}{2} \sum_{j=1}^r (q_i E_{2e_j} + r_i E_{-2e_j}), \quad (40)$$

while in the D_r -case we have;

$$Q(x, t) = \sum_{i < j} (q_{ij}E_{e_i+e_j} + r_{ij}E_{-e_i-e_j}), \quad (41)$$

In the typical representations of C_r and D_r these choices for $Q(x, t)$ have always the block structure shown in (10). In the case of $\mathfrak{g} \simeq sp(4)$ the blocks \mathbf{q} and \mathbf{r} are parametrized by three functions each:

$$\mathbf{q}(x, t) = \begin{pmatrix} q_{12} & q_1 \\ q_2 & q_{12} \end{pmatrix}, \quad \mathbf{r}(x, t) = \begin{pmatrix} r_{12} & r_2 \\ r_1 & r_{12} \end{pmatrix}, \quad (42)$$

while for $\mathfrak{g} \simeq so(8)$ they contain six independent functions each:

$$\mathbf{q}(x, t) = \begin{pmatrix} q_{14} & q_{13} & q_{12} & 0 \\ q_{24} & q_{23} & 0 & q_{12} \\ q_{34} & 0 & q_{23} & -q_{13} \\ 0 & q_{34} & -q_{24} & q_{14} \end{pmatrix}, \quad \mathbf{r}(x, t) = \begin{pmatrix} r_{14} & r_{24} & r_{34} & 0 \\ r_{13} & r_{23} & 0 & r_{34} \\ r_{12} & 0 & r_{23} & -r_{24} \\ 0 & r_{12} & -r_{13} & r_{14} \end{pmatrix}, \quad (43)$$

The corresponding sets of MNLS eqs. (5) for these two choices of $Q(x, t)$ and for VBC were first derived in [6]. For CBC with $\mathbf{r} = \mathbf{q}^\dagger$ they take the form (15) with the additional linear in \mathbf{q} terms ensuring regular behavior for $t \rightarrow \pm\infty$.

Note that reducing $Q(x, t)$ to take values in \mathfrak{g} then we naturally have that $\psi_0(\lambda)$ and $\phi_0(\lambda)$ take values in the corresponding group \mathfrak{G} .

4.1 Spectral properties of $sp(4)$ -MNLS with CBC

As mentioned in Section 3, the continuous spectrum of the GZS system (8) is determined by the set of eigenvalues $\{\nu_j, j = 1, 2\}$ of the matrices $q_+ r_+ = q_- r_-$. These eigenvalues for $Q(x, t)$ given by eqs. (40), (42) with $r = 2$ satisfy the characteristic equation:

$$\nu^2 - K_0 \nu + K_1 = 0, \quad K_0 = \frac{1}{2} \text{tr } Q_\pm^2, \quad K_1 = \det Q_\pm. \quad (44)$$

and determine the end points of the spectrum. If we impose on $Q(x, t)$, and consequently on Q_\pm the involution (\mathbb{Z}_2 -reduction):

$$B_1^{-1} Q^\dagger B_1 = Q, \quad B_1 = \text{diag}(1, \epsilon, \epsilon, 1), \quad \epsilon = \pm 1. \quad (45)$$

which in components takes the form:

$$r_1 = \epsilon q_1^*, \quad r_2 = q_2^*, \quad r_3 = q_3^*. \quad (46)$$

Then the coefficients K_0 and K_1 equal:

$$K_0 = 2\epsilon |q_1^\pm|^2 + |q_2^\pm|^2 + |q_3^\pm|^2, \quad K_1 = |(q_1^\pm)^2 + q_2^\pm q_3^\pm|^2 \quad (47)$$

We have three possibilities for the roots ν_1, ν_2 of eq. (44) depending on the sign of the discriminant:

$$D = \frac{1}{4} K_0^2 - 4K_1. \quad (48)$$

- a) $D > 0$, i.e. the roots $\nu_1 > \nu_2$ are different and real. The continuous spectrum of L fills up two pairs of rays on the real axis $|\text{Re } \lambda| > \nu_1$ and $|\text{Re } \lambda| > \text{Re } \nu_2$;
- b) $D = 0$, i.e. the roots $\nu_1 = \nu_2$; the two pairs of rays in a) now coincide; the total multiplicity of the spectrum is 4;
- c) $D < 0$, i.e. the roots ν_j are complex-valued and $\nu_1 = \nu_2^*$; The continuous spectrum of L fills up two branches of two-fold spectrum along the hyperbola's arcs $\text{Re } \lambda \text{Im } \lambda = \text{Re } \nu_k \text{Im } \nu_k$, see the right panel of fig. 1;

In the generic case there are no apriory limitations as to the positions of the discrete eigenvalues. Such may come up if we consider potentials $Q = -Q^\dagger$; then the GZS system become equivalent to a formally self-adjoint linear problem whose spectrum should be confined to the real λ -axis only. The formal self-adjointness takes place for $\epsilon = 1$.

4.2 Spectral properties of $so(8)$ -MNLS with CBC

The characteristic equation for $q_{\pm}r_{\pm}$ takes more simple form:

$$\det(q_{\pm}r_{\pm} - \nu) = (\nu^2 - K_0\nu + K_1)^2, \quad (49)$$

where the coefficients K_j now are given by:

$$K_0 = \frac{1}{2} \text{tr}(q_{\pm}r_{\pm}) = \sum_{1 \leq i < j \leq 4} q_{ij}^{\pm} r_{ij}^{\pm}, \quad (50)$$

$$K_1 = (\det(q_{\pm}r_{\pm}))^{1/2} = (q_{13}^{\pm}q_{24}^{\pm} - q_{34}^{\pm}q_{12}^{\pm} - q_{23}^{\pm}q_{14}^{\pm})(r_{13}^{\pm}r_{24}^{\pm} - r_{34}^{\pm}r_{12}^{\pm} - r_{23}^{\pm}r_{14}^{\pm}).$$

An involution of the type (45) gives $r_{ij} = \epsilon_i \epsilon_j q_{ij}^*$ with $\epsilon_j = \pm 1$ and makes the coefficients K_0, K_1 real. Besides now each of the eigenvalues ν_j , $j = 1, 2$ is two-fold. Again we have the three possibilities depending on the value of D ; the only difference is that the multiplicity of each of the branches is 4. This imposes certain symmetry on the locations of the eigenvalues of ν_j which in fact determine the end-points of the continuous spectra of L .

5 Hamiltonian properties

The invariants of the transfer matrix $T(\lambda)$ such as, e.g. its principal minors generate integrals of motion, i.e. if all ν_j are different we have only r independent series of conserved quantities. Let us briefly outline the methods of deriving of these integrals as functionals of the potential Q . As starting relation here we consider the Wronskian relation, generalizing the one derived in [17] for the scalar case $s = 1$:

$$\begin{aligned} & \text{tr} \left[i(\chi^+)^{-1} \frac{d\chi^+}{d\lambda} C - \frac{dJ(\lambda)}{d\lambda} x C \right] \Big|_{x=-\infty}^{\infty} \\ &= i \sum_{k=1}^r \frac{d\delta_k^+}{d\lambda} \text{tr}(H_k C) + i \text{tr} \left(\hat{\psi}_0(\lambda) \frac{d\psi_0}{d\lambda} C - \hat{\phi}_0(\lambda) \frac{d\phi_0}{d\lambda} C \right) \\ &= \int_{-\infty}^{\infty} dx \text{tr} [\sigma_3 R_C(x, t, \lambda) - \lambda J^{-1}(\lambda) C], \end{aligned} \quad (51)$$

where C is a constant element of \mathfrak{h} and $R_C(x, t, \lambda) = \chi^+ C (\chi^+)^{-1}(x, t, \lambda)$ is a natural generalization of the diagonal of the resolvent of the system (8). It satisfies the equation:

$$i \frac{dR_C}{dx} + [Q(x, t) - \lambda J, R_C] = 0, \quad \lim_{x \rightarrow \infty} R_C(x, t, \lambda) = \psi_0(\lambda) C \psi_0^{-1}(\lambda). \quad (52)$$

Eq. (52) allows one to derive the recurrent relations for evaluating the expansion coefficients

$$R_C(x, t, \lambda) = C_0 + \sum_{k=1}^{\infty} R_k \lambda^{-k}(x, t), \quad \psi_0 C \psi_0^{-1}(\lambda) = C_0 + \sum_{k=1}^{\infty} C_k \lambda^{-k}. \quad (53)$$

The trace identities for the MNLS type equations with CBC can be derived by inserting the asymptotic expansions of $R_C(x, \lambda)$ and $\delta_k^+(\lambda)$:

$$\delta_k^+(\lambda) = \sum_{p=1}^{\infty} I_p^{(k)} \lambda^{-p} \quad (54)$$

in both sides of eq. (51) and equating the corresponding coefficients of λ^{-p} .

Here we write down the first three of the local integrals of motion coming from the principal series with $C = J$:

$$\begin{aligned} \mathbf{H}_k &= \int_{-\infty}^{\infty} dx \operatorname{tr} (\sigma_3 R_{k+1}(x, t) - C_{k+1}), & \mathbf{H}_1 &= \frac{1}{2} \int_{-\infty}^{\infty} dx \operatorname{tr} (qq^\dagger(x, t) - \bar{\mu}), \\ \mathbf{H}_2 &= \frac{i}{4} \int_{-\infty}^{\infty} dx \operatorname{tr} (q_x q^\dagger - q q_x^\dagger), & \bar{\mu} &= q_+ q_+^\dagger, \\ \mathbf{H}_3 &= \frac{3}{8} \int_{-\infty}^{\infty} dx \operatorname{tr} [q_x q_x^\dagger + (qq^\dagger(x, t))^2 - \bar{\mu}^2]. \end{aligned} \quad (55)$$

The correct use of the Wronskian relation (51) allowed us to derive renormalized integrals of motion, i.e. ones that converge for $Q(x, t) \in \mathcal{M}$.

However among the integrals in this series one can not find the Hamiltonian of the MNLS (15). In order to obtain the Hamiltonian we need to regularize these integrals. By regularized integral we mean one whose gradient $\delta \mathbf{H}_k / \delta Q^T(x, t)$ vanishes for both $x \rightarrow \infty$ and $x \rightarrow -\infty$. This can be done by considering additional series of integrals, which generically have non-local densities. Fortunately among the simplests of them one may find local ones. For example, the first integral from the series with C choosen to be $C^{(l)} = \sum_{k=1}^r m_k^{2l} H_k$ with $1 \leq l \leq r$, is local and has the form:

$$\tilde{\mathbf{H}}_1^{(l)} = \frac{1}{4} \int_{-\infty}^{\infty} dx \operatorname{tr} [qq^\dagger(x, t) \bar{\mu}^l + q^\dagger q(x, t) \mu^l - 2\mu^{l+1}], \quad \mu = q_+^\dagger q_+. \quad (56)$$

Note, that $\tilde{\mathbf{H}}_1^{(l)}$ is nontrivial, i.e. does not reduce to \mathbf{H}_1 only if $s \geq 2$, $\nu_1 \neq \nu_2 \neq \dots$. Using it we can check the validity of

$$\mathbf{H}_{\text{MNLS}} = \frac{8}{3} \mathbf{H}_3 - 4 \tilde{\mathbf{H}}_1^{(1)} = \int_{-\infty}^{\infty} dx \operatorname{tr} [q_x q_x^\dagger + (qq^\dagger(x, t) - \bar{\mu})^2]. \quad (57)$$

In analyzing the Hamiltonian properties of the MNLS with CBC we will make use also of the classical r -matrix approach, see [4, 19]. It allows one to write down in compact form the Poisson brackets of the transfer (monodromy) matrix. Since our problem is ultra-local in the terminology of [4, 19] then the definition of r is independent on the boundary conditions. Taking into account the results of [6] we find

$$r(\lambda, \mu) = \frac{1}{\lambda - \mu} \left(\sum_{\alpha \in \Delta_0^+ \cup \Delta_1^+} (E_\alpha \otimes E_{-\alpha} + E_{-\alpha} \otimes E_\alpha) + \sum_{j=1}^r H_j \otimes H_j \right), \quad (58)$$

where E_α and H_j are the Cartan-Weyl generators of the corresponding simple Lie algebra \mathfrak{g} , see [24]. In order to derive the Poisson brackets for the MNLS on the whole axis with CBC we need to take into account the corresponding oscillations coming from the Jost solutions.

Skipping the details we just write down the expressions for the Poisson brackets between the matrix elements of $T(\lambda)$:

$$\begin{aligned} \left\{ T(\lambda) \otimes T(\mu) \right\} &= r_+(\lambda, \mu) T(\lambda) \otimes T(\mu) - T(\lambda) \otimes T(\mu) r_-(\lambda, \mu), \quad (59) \\ r_\pm(\lambda, \mu) &= \lim_{x \rightarrow \pm\infty} \tau_\pm^{-1}(x, \lambda, \mu) r(\lambda, \mu) \tau_\pm(x, \lambda, \mu), \\ \tau_+(x, \lambda, \mu) &= \psi_0(\lambda) e^{-iJ(\lambda)x} \otimes \psi_0(\mu) e^{-iJ(\mu)x}, \\ \tau_-(x, \lambda, \mu) &= \phi_0(\lambda) e^{-iJ(\lambda)x} \otimes \phi_0(\mu) e^{-iJ(\mu)x}, \end{aligned}$$

where $\{T(\lambda) \otimes T(\mu)\}_{ij,kl} \equiv \{T_{ij}(\lambda), T_{kl}(\mu)\}$.

Similar problem for the scalar case was analyzed in [17, 19]. For $\mathfrak{g} \simeq C_r$, or D_r the explicit form of $r_\pm(\lambda, \mu)$ is rather complicated and will be published elsewhere. An important and difficult problem here is to take correctly into account the threshold singularities of $T_{kl}(\lambda)$ of the form $j_k^{-1}(\lambda)$ at the end points of the continuous spectrum.

An important consequence of (59) are the involution properties of $\delta_k^+(\lambda)$:

$$\{\delta_k^+(\lambda), \delta_j^+(\mu)\} = 0, \quad (60)$$

for all values of $1 \leq i, j \leq r$ and λ and μ taking values on the continuous spectrum of L . From (60) there follows that $\{I_p^{(k)}, I_s^{(l)}\} = 0$ for all positive values of p and s , and for all $1 \leq k, l \leq r$. A consequence of eq. (60) is the involutivity of the integrals of the principal series $\{\mathbf{H}_k, \mathbf{H}_p\} = 0$. This is a necessary condition in proving the complete integrability of the MNLS equations with CBC; other difficulties in proving it are outlined in [9].

Of course the rigorous proof of the complete integrability and the derivation of the basic properties of the MNLS equations must be based on the completeness relation of the relevant ‘squared solutions’ of L . For the single component NLS such relation has been proposed in [20]; for the multicomponent systems this is still open question.

6 Discussion

The $sp(4)$ MNLS can be viewed as a special reduction from the $su(4)$ one. Recently it was discovered that the $sp(4)$ MNLS with vanishing BC has important applications to BEC [25]. This enhances the interest to the MNLS type models. In particular it will be important to work out the dressing Zakharov-Shabat procedure not only for MNLS with vanishing boundary conditions, but also for non-vanishing BC. Some steps in this directions have been reported in [18] for the $su(n+m)/s(u(n) \otimes su(m))$ case. Deriving the dressing factors for the MNLS

for the symmetric spaces of types *C.II*- and *D.III*- requires substantial changes even for vanishing BC; doing the same for CBC is still a bigger challenge.

Similar methods can be applied to the analysis of the *N*-wave type equations with CBC, see [21].

Acknowledgements

This work has been supported by the National Science Foundation of Bulgaria, contract No. F-1410.

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